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Quantum Mechanically Induced Hopf Term in the O(3) Nonlinear Sigma Model

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Abstract. The Hopf term in the $2+1$ dimensional $O(3)$ nonlinear sigma model, which is known to be responsible for fractional spin and statistics, is re-examined from the viewpoint of quantization ambiguity. It is confirmed that the Hopf term can be understood as a term induced quantum mechanically, in precisely the same manner as the θ-term in QCD. We present a detailed analysis of the topological aspect of the model based on the adjoint orbit parametrization of the spin vectors, which is not only very useful in handling topological (soliton and/or Hopf) numbers, but also plays a crucial role in defining the Hopf term for configurations of nonvanishing soliton numbers. The Hopf term is seen to arise explicitly as a quantum effect which emerges when quantizing an S^1 degree of freedom hidden in the configuration space.

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1. Introduction

The $O(3)$ nonlinear sigma model (NLSM) is a very useful model, almost ubiquitous in physics, appearing in various circumstances where the original $O(3)$ symmetry of the system is spontaneously broken. In condensed matter physics, it describes systems ranging from anti-ferromagnetic spin chains to a certain class of materials which exhibit quantum Hall effects. In particle physics, it is considered to be a prototype of QCD, primarily because the (pure) NLSM is asymptotically free in $D = 1 + 1$ dimensions. Classically, the model is governed by the action

$$I = \int d^D x \, \frac{1}{2\lambda^2} \, \partial_\mu \mathbf{n}(x) \cdot \partial^\mu \mathbf{n}(x), \tag{1.1}$$

where λ is a coupling constant and $\mathbf{n}(x) = (n_1(x), n_2(x), n_3(x))$ is a field of spin vectors which are subject to the constraint $\mathbf{n}^2(x) = \sum_a n_a^2(x) = 1$. Under appropriate boundary conditions, topological terms $I_{\rm top}$ of strength θ (which is an angle parameter) are available, and can be added to the action I in (1.1). In $1 + 1$ dimensions, the topological term is given by [1]

$$I_{\rm top} = \theta \int d^2 x \, \frac{1}{8\pi} \epsilon^{ij} \epsilon^{abc} \, n_a(x) \, \partial_i n_b(x) \, \partial_j n_c(x). \tag{1.2}$$

It is known [2] that the $O(3)$ NLSM can be derived as an effective low energy theory for the anti-ferromagnetic Heisenberg spin chains, where the angle parameter turns out to be $\theta = 2\pi S$ for spin S chains. This result is consistent with experimental observations [3].

Meanwhile, in $D = 2 + 1$ dimensions, we can define the topological current,

$$J^\mu = \frac{1}{8\pi} \epsilon^{\mu\nu\lambda} \epsilon^{abc} n_a \partial_\nu n_b \partial_\lambda n_c, \tag{1.3}$$

together with the gauge potential A_μ from the relation $J^\mu = \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda$, and thereby furnish the topological term [4,5] (see also [6]),

$$I_{\rm top} = -\theta \int d^3 x \, A_\mu(x) \, J^\mu(x) = \theta \, H(\mathbf{n}). \tag{1.4}$$

The term $H(\mathbf{n})$, called the Hopf (invariant) term, is nonlocal when expressed in terms of $\mathbf{n}(x)$, but can be made local by introducing, for instance, the $\mathbb{C}P^1$ parametrization [5] of the spin vectors. In contrast to the $1 + 1$ dimensional case, the topological term does not arise (*i.e.*, $\theta = 0$) in the effective theory of the spin chains [7,8,9], and this result is also experimentally supported [10]. However, it is argued [11] that the Hopf term does emerge

in a slightly different situation, that is, in the spin S ladders (a system consisting of n_l ladders of spin chains coupled weakly) where the angle parameter is given by $\theta = 2\pi n_l S$. A nonvanishing Hopf term has also been reported to arise with the value $\theta = \pi$ in the NLSM derived in describing the quantum Hall effect at small Zeeman energies [12,13,14].

The addition of these topological terms has major ramifications for the physics. For example, in $1+1$ dimensions it is believed [2] that, unlike for the case of integral spin S, the spectrum becomes gapless for half-integral S because the NLSM with the angle $\theta = \pi$ acquires a fixed point with vanishing mass in renormalization group flow, at which it becomes equivalent to the $SU(2)$ $k = 1$ Wess-Zumino-Novikov-Witten model [15]. Moreover, in $2+1$ dimensions, it has been shown [4,16] that fractional spin and statistics may occur if the angle θ is nonvanishing.

On the other hand, we know from the general theory of quantization (see, *e.g.*, [17]) that topological terms can be induced upon quantization, if the configuration space $\mathcal{Q}$ of the system is topologically nontrivial. A familiar case of such configuration spaces arises when the space $\mathcal{Q}$ is multiply connected [18,19,20], namely, when the fundamental group of the space $\pi_1(\mathcal{Q})$ is nontrivial. This case admits the appealing interpretation that induced terms derive from the ambiguity in phase in defining the path-integral (although a similar interpretation is also possible for more general cases [21]), with the phase being given by a unitary representation of the fundamental group $\pi_1(\mathcal{Q})$. The simplest example of this case is the circle $\mathcal{Q} = S^1$ for which $\pi_1(\mathcal{Q}) = \mathbb{Z}$. Parametrizing the circle by an angle ϕ, we find, at the quantum level, the induced topological term [22],

$$I_{\rm top} = \theta \int dt \, \frac{1}{2\pi} \frac{d\phi}{dt}. \tag{1.5}$$

The parameter θ signifies the possible inequivalent quantizations (or superselection sectors), which can be determined only by extra physical requirements, such as the amount of magnetic flux penetrating the circle if the system is put in the setting of the Aharonov-Bohm effect. If such a requirement is not available, the parameter is intrinsically indeterminable, as in the case of the Yang-Mills theory where the θ-term causes the strong CP problem.

In fact, the topological terms in the $O(3)$ NLSM are the analogue of the θ-term in the Yang-Mills theory, since they can be understood as quantum mechanically induced terms in the sense mentioned above. Thus it needs to be examined whether or not those topological terms pose a similar problem in the $O(3)$ NLSM, too. Discussions from the viewpoint of the

analogy with the Yang-Mills theory have been given in ref.[23] using the phase ambiguity interpretation of the path-integral, which is available since the configuration space $\mathcal{Q}$ of the NLSM has $\pi_1(\mathcal{Q}) = \mathbb{Z}$. The aim of the present paper is to provide a different argument for this on a firm basis, rather than using the phase ambiguity interpretation. Specifically, we shall uncover the fact that, in the $2+1$ dimensional $O(3)$ NLSM, it is the S^1 degree of freedom hidden in $\mathcal{Q}$ that induces the Hopf term (1.4) in precisely the form of the induced term (1.5) at the quantum level. In the course of the discussion we employ the adjoint orbit parametrization of the spin vector field $\mathbf{n}(x)$ in terms of an $SU(2)$ group valued field $g(x)$. This parametrization, which has been advocated in refs.[24,25], not only renders the Hopf term local as the $\mathbb{C}P^1$ parametrization does, but also provides a very convenient tool in analyzing the topological aspects of the configurations we need to consider. We present here a more rigorous treatment of the adjoint orbit parametrization than previously given, so that the Hopf term can be dealt with properly in the NLSM for configurations of any soliton numbers.

This paper is organized as follows: after this Introduction, in sect.2 we introduce the adjoint orbit parametrization for the spin vectors in the NLSM, paying special attention to the validity of the parametrization in regard to the soliton numbers. Then in sect.3 we define the soliton and Hopf numbers to be assigned to a generic configuration based on the adjoint orbit parametrization, which furnishes a coherent framework for dealing with topologically nontrivial configurations. Sect.4 contains an important statement about the decomposition of a generic configuration, which allows us to extract the S^1 degree of freedom from $\mathcal{Q}$ and induce the Hopf term as mentioned. Finally, sect.5 is devoted to discussions and outlooks. An appendix is supplied at the end to provide a mathematical account of the topological invariants and homotopy groups discussed in the text.

2. Adjoint orbit parametrization

The parametrization of spin vectors in terms of an adjoint orbit of $SU(2)$, or more generally the parametrization of vectors taking values on a coset space G/H in terms of an adjoint orbit of the group G, was introduced in ref.[24] in an attempt to formulate the NLSM as a gauge theory. In the $O(3)$ NLSM, we take the coset to be $G/H = SU(2)/U(1) \simeq S^2$, where $H = U(1)$ is the subgroup generated by, say, the element T_3 in the basis $\{T_a = \frac{\sigma_a}{2i}; a = 1,2,3\}$ of the Lie algebra of $SU(2)$. In more detail, given a field $\mathbf{n}(x)$ taking

values on a sphere S^2, we consider a field $g(x)$ which takes values on the group manifold $G = SU(2)$ and reproduces the spin vectors $\mathbf{n}(x)$ by the formula,

$$g(x)\, T_3\, g^{-1}(x) = n(x) := n_1(x)T_1 + n_2(x)T_2 + n_3(x)T_3. \tag{2.1}$$

In other words, the adjoint orbit parametrization consists of identifying the target space S^2 of the spin vectors with the adjoint orbit $\mathcal{O}_K$ of $SU(2)$ passing through the element K which we choose to be T_3. In what follows we adopt the convention $\mathrm{Tr} := (-2)$ times the matrix trace, which leads to the normalized trace $\mathrm{Tr}\,(T_a T_b) = \delta_{ab}$. Then it is seen that the constraint satisfied by the spin vectors reads $\mathrm{Tr}\, n^2(x) = 1$, and that this condition is automatically fulfilled by the parametrization (2.1). Note, however, that the adjoint orbit parametrization possesses redundancy, since $\mathbf{n}(x)$ is unchanged under the 'gauge transformations',

$$g(x) \longrightarrow g(x)h(x), \qquad \text{where} \quad h(x) \in H = U(1). \tag{2.2}$$

We shall see later that the gauge transformations are well-defined only for topologically trivial functions $h(x)$.

In order to discuss the class of maps that can be used for $g(x)$, we now specify the class of configurations of spin vectors $\mathbf{n}(x)$ in the NLSM which we will be interested in. We first take our spacetime to be $\mathbb{R}^2 \times [0, T]$. Then the configurations of interest are those which become a single constant vector at infinity in space,

$$\mathbf{n}(x) = \mathbf{n}(\mathbf{x}, t) \longrightarrow \mathbf{n}(\infty) \qquad \text{as} \quad \|\mathbf{x}\| \to \infty. \tag{2.3}$$

The vector $\mathbf{n}(\infty)$ is assumed to be constant not only at infinity in space $\|\mathbf{x}\| \to \infty$ for $\mathbf{x} \in \mathbb{R}^2$ but also in time $t \in [0, T]$. Then, by adding a point, say, the South pole $\mathbf{x}_{\mathrm{S}}$ to the space $\mathbb{R}^2$ and identifying it with a sphere S^2, this allows us to regard $\mathbf{n}(x)$ as a map $S^2 \times [0, T] \to S^2$ with the value at the South pole $\mathbf{n}(\mathbf{x}_{\mathrm{S}}, t) = \mathbf{n}(\infty)$ fixed for any $t \in [0, T]$. The space of these maps, *i.e.*, based maps from S^2 to S^2, is the configuration space $\mathcal{Q}$ of our $O(3)$ NLSM,

$$\mathcal{Q} = \mathrm{Map}_0(S^2, S^2). \tag{2.4}$$

A salient feature of the space of based maps is that it satisfies the following useful identity for homotopy groups (see Appendix),

$$\pi_k(\mathcal{Q}) = \pi_{k+2}(S^2), \qquad \text{for} \quad k = 0, 1, 2, \ldots. \tag{2.5}$$

Thus, in particular, we have

$$\pi_0(\mathcal{Q}) = \pi_2(S^2) = \mathbb{Z}, \tag{2.6}$$

which states that the configuration space $\mathcal{Q}$ is disconnected into 'n-soliton sectors' $\mathcal{Q}_n$ labelled by the integers n which correspond to $\pi_2(S^2)$:

$$\mathcal{Q} = \bigcup_{n\in\mathbb{Z}} \mathcal{Q}_n. \tag{2.7}$$

We also have

$$\pi_1(\mathcal{Q}) = \pi_3(S^2) = \mathbb{Z}, \tag{2.8}$$

which implies that each soliton sector is multiply connected $\pi_1(\mathcal{Q}_n) = \mathbb{Z}$, hence providing the basis for the existence of the Hopf term (1.4). The basic motivation for employing the adjoint orbit parametrization is to gain a better control over this topological structure of the configuration space by making use of the group property.

Returning to the field $g(x)$, the first question we need to address is the very existence of the map $S^2 \times [0,T] \to SU(2) \simeq S^3$ that will be assigned to $g(x)$ if it is to correspond under (2.1) to the map $S^2 \times [0,T] \to S^2$ given by $\mathbf{n}(x)$. The answer is negative, however, because if there was such a map then there would also be a map $S^2 \to S^3$ (from the target S^2 to the target S^3 at a fixed time), but this is impossible since the S^3, regarded as a $U(1)$ principal bundle over the base space S^2, is known to be nontrivial in general. (For more detail, see Appendix.) An obvious wayout is to trivialize the bundle, that is, first remove the South pole and consider the map $D^2 \times [0,T] \to S^2$ for $\mathbf{n}(x)$, where D^2 is a two dimensional disc of unit radius which is identified with $S^2 - \{\mathbf{x}_\mathrm{S}\}$, and then find a map $D^2 \times [0,T] \to S^3$ for $g(x)$ related to $\mathbf{n}(x)$ under (2.1). When this is done, the map $g(x)$ must reproduce the constant spin vector $\mathbf{n}(\infty)$ on the boundary of the disc $\partial D^2 = S^1$, that is,

$$g(x)\, T_3\, g^{-1}(x) = n(\infty), \qquad \text{for} \quad x \in \partial D^2 \times [0,T]. \tag{2.9}$$

If we let $g(\infty)$ be a representative fixed element fulfilling (2.9), then, due to the ambiguity under (2.2), we find that $g(x)$ takes the following form on the boundary,

$$g(x) = g(\infty)h(x), \qquad h(x) \in H = U(1), \qquad \text{for} \quad x \in \partial D^2 \times [0,T], \tag{2.10}$$

where $h(x)$ is an arbitrary function of space and time. At this point we remark that the function $h(x)$ in (2.10), which characterizes the field $g(x)$ on the boundary ∂D^2, can be

nontrivial as a map $S^1 \to U(1)$, whereas those functions $h(x)$ used for gauge transformations (2.2) must be trivial because they define a $U(1)$ bundle over the disc D^2, which is trivial since D^2 is contractible.

To sum up the foregoing argument in terms of spaces, consider first the space of maps for $g(x)$,

$$\mathcal{G} = \mathrm{Map}'(D^2, SU(2)), \tag{2.11}$$

where the prime indicates that maps in $\mathcal{G}$ fulfill the condition (2.10). Let now

$$\mathcal{H} = \mathrm{Map}(D^2, U(1)), \tag{2.12}$$

be the space of functions $h(x)$ used for gauge transformations (2.2). Then, the adjoint orbit parametrization amounts to identifying the configuration space $\mathcal{Q}$ of the spin vectors $\mathbf{n}(x)$ given in (2.4) with the quotient space,

$$\mathcal{Q} = \mathcal{G}/\mathcal{H}, \tag{2.13}$$

where $\mathcal{G}/\mathcal{H}$ is the orbit space of $\mathcal{G}$ under $\mathcal{H}$, *i.e.*, the set of equivalent classes under gauge transformations (2.2).

An important point to note is that the space $\mathcal{G}$, which is characterized by the boundary value $g(\infty)$ on ∂D^2, does not in general form a group. In fact, multiplication may not be well-defined in $\mathcal{G}$ since it does not necessarily respect the boundary condition (2.10), or the inverse g^{-1} may not exist in $\mathcal{G}$ since g^{-1} may not obey (2.10) even if g does. There are, however, two cases of $\mathcal{G}$ worth noticing. One is the case where $g(\infty) \in H$ on ∂D^2. In this case, the space $\mathcal{G}$, which we denote by $\mathcal{G}^+$, does form a group, enjoying the property $g^+ g^{+\prime} \in \mathcal{G}^+$ and $(g^+)^{-1} \in \mathcal{G}^+$ for any g^+, $g^{+\prime}$ in $\mathcal{G}^+$. A remarkable property of the space $\mathcal{G}^+$ is that it preserves any space $\mathcal{G}$ under right-multiplication,

$$g\, g^+ \in \mathcal{G} \qquad \text{for} \quad g \in \mathcal{G}, \quad g^+ \in \mathcal{G}^+. \tag{2.14}$$

The other space, which we denote by $\mathcal{G}^-$, occurs when $g(\infty) \in e^{\pi T_2} H$ on ∂D^2. Although the space $\mathcal{G}^-$ does not itself form a group, it satisfies $(g^-)^{-1} \in \mathcal{G}^-$ for g^- in $\mathcal{G}^-$ and

$$g^- g^{-\prime} \in \mathcal{G}^+, \qquad g^- g^+ \in \mathcal{G}^-, \qquad g^+ g^- \in \mathcal{G}^-, \tag{2.15}$$

where we have used $T_3 e^{\pi T_2} = -e^{\pi T_2} T_3$. For this reason we may assign '±' ('even' and 'odd') parity to $\mathcal{G}^\pm$, respectively. These spaces $\mathcal{G}^\pm$ are actually homeomorphic to each other under the parity operation[4]

$$g \longrightarrow e^{\pi T_2}\, g. \tag{2.16}$$

[4] Notice the difference from the spin flip operation $\mathbf{n}(x) \to -\mathbf{n}(x)$ implemented by $g \to g\, e^{\pi T_2}$.

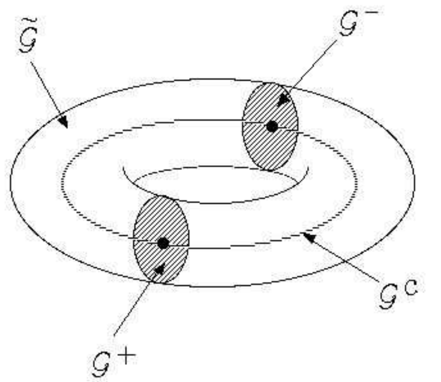

Figure 1. A schematic picture of the spaces $\mathcal{G}^+$, $\mathcal{G}^-$ and $\mathcal{G}^{\rm c}$ in $\widetilde{\mathcal{G}}$ (which is represented by a doughnut $D^2 \times S^1$). Each of the two intersection points in $\mathcal{G}^{\rm c}$ actually consists of a circle S^1; one with $\mathcal{G}^+$ is given by H and the other with $\mathcal{G}^-$ by $e^{\pi T_2} H$.

We also mention that, in addition to these two spaces, there is a space $\mathcal{G}^{\rm c}$, which too forms a group, consisting of configurations which become constant $g(x) = g_{\rm c}$ on ∂D^2. If we define the space of all configurations of the type (2.10) with (not a fixed but) any $g(\infty) \in SU(2)$,

$$\widetilde{\mathcal{G}} = \bigcup_{g(\infty)\in SU(2)} \mathcal{G}, \tag{2.17}$$

then we find that in $\widetilde{\mathcal{G}}$ the subspace $\mathcal{G}^{\rm c}$ intersects with the other subspaces $\mathcal{G}^+$ and $\mathcal{G}^-$ at $g_{\rm c} = 1$ and $g_{\rm c} = e^{\pi T_2}$ modulo $H = U(1)$ as shown in Fig.1.

Let us now consider under what circumstances multiplication is well-defined in the total space $\widetilde{\mathcal{G}}$. To this end, note first that the condition $g \in \widetilde{\mathcal{G}}$ is equivalent to

$$g^{-1}dg \in \mathfrak{u}(1), \qquad \text{for} \quad x \in \partial D^2 \times [\,0, T\,], \tag{2.18}$$

where $\mathfrak{u}(1)$ is the Lie algebra of $H = U(1)$ generated by T_3. Thus, for the product $g\,g'$ formed from g, $g' \in \widetilde{\mathcal{G}}$ to belong again to $\widetilde{\mathcal{G}}$ we must have $(g\,g')^{-1}d(g\,g') \in \mathfrak{u}(1)$, but this holds if and only if we have on the boundary,

$$g'(\infty)^{-1}\,(g^{-1}dg)\,g'(\infty) \in \mathfrak{u}(1), \tag{2.19}$$

where $g'(\infty)$ is the constant fixed element that characterizes g' on the boundary ∂D^2. Parametrize the element $g'(\infty)$ by the Euler angles $(\rho, \varphi, \chi) \in [0, 2\pi] \times [0, \pi] \times [0, 4\pi]$ as

$$g'(\infty) = e^{\rho T_3}\, e^{\varphi T_2}\, e^{\chi T_3}. \tag{2.20}$$

Since $g^{-1}dg \in \mathfrak{u}(1)$ on ∂D^2, we can write $g^{-1}dg = \gamma_3 T_3$ with a 1-form γ_3. Then, a straightforward computation yields

$$g'(\infty)^{-1}\,(g^{-1}dg)\,g'(\infty) = \gamma_3\left[\cos\varphi\, T_3 - \sin\varphi\,(\cos\chi\, T_1 - \sin\chi\, T_2)\right]. \tag{2.21}$$

We thus find that, for (2.19) to be valid, we need either (i) $\gamma_3 = 0$, (ii) $\varphi = 0$ or (iii) $\varphi = \pi$. Conversely, (2.19) holds if any of (i) – (iii) holds. Observe that (i) is the case where $g \in \mathcal{G}^{\mathrm{c}}$, whereas (ii) and (iii) imply $g' \in \mathcal{G}^{+}$ and $g' \in \mathcal{G}^{-}$, respectively.

In conclusion, multiplication in $\widetilde{\mathcal{G}}$ is well-defined if and only if (at least) one of the two elements belong to those subspaces in the manner stated above. Note also that the space $\mathcal{G}$ is not necessarily preserved under multiplication, except for case (ii) which is the right-multiplication by $\mathcal{G}^{+}$ mentioned earlier. For instance, gauge transformations (2.2) preserve the space since they belong to case (ii), whereas global $SU(2)$ left-transformations,

$$g(x) \longrightarrow g_{\mathrm{c}}\, g(x), \qquad \text{where} \quad g_{\mathrm{c}} \in SU(2), \tag{2.22}$$

do not, since they belong to case (i). The latter transformations lead to $O(3)$ rigid rotations of the vectors $\mathbf{n}(x)$ through (2.1), which are the symmetry transformations of the action (1.1).

3. Topological (soliton and Hopf) numbers

The advantage of using the adjoint orbit parametrization becomes manifest when we deal with the soliton number which is the charge of the topological current (1.3),

$$Q(\mathbf{n}) = \int_{S^2} d^2x\, J^0(x). \tag{3.1}$$

The soliton number $Q(\mathbf{n})$ is actually the wrapping number of the map $S^2 \to S^2$ given by $\mathbf{n}(x)$ at any fixed time, and hence takes an integer related to the homotopy group (2.6). On the other hand, the soliton number is introduced also to $g(x)$ by

$$Q(g) = -\frac{1}{4\pi}\int_{\partial D^2} \mathrm{Tr}\, T_3(g^{-1}(x)dg(x)). \tag{3.2}$$

Indeed, changing the domain of the integral in (3.1) to D^2 by removing the South pole $\mathbf{x}_{\mathrm{S}}$ from S^2 (which does not change the outcome of the integral) and using the relation (2.1), we can easily confirm that the expression $Q(\mathbf{n})$ in (3.1) reduces to $Q(g)$ in (3.2). It then

follows from (2.10) that the soliton number (3.2) is nothing but the winding number of the map $\partial D^2 \simeq S^1 \to H = U(1)$ given by $h(x)$ on the boundary ∂D^2. In other words, the piece $h(x)$ in $g(x)$ in (2.10) which corresponds to a given spin vector field $\mathbf{n}(x)$ is chosen such that its winding number be equal to the soliton number $Q(\mathbf{n})$ and thereby relates the two integers, $\pi_1(U(1)) = \mathbb{Z}$ and $\pi_2(S^2) = \mathbb{Z}$. As we remarked earlier, gauge transformations (2.2) are trivial on the boundary ∂D^2 as a map $S^1 \to U(1)$ and cannot change the winding number, that is, the soliton number $Q(g)$ is gauge invariant, as required. We also note that the zero soliton sector $\mathcal{Q}_0$ is the only sector where the $U(1)$ bundle over $\partial D^2 \simeq S^1$ is trivial and hence $h(x)$ can be chosen to be constant on the boundary, that is, the sector where $g(x)$ is well-defined as a map $S^2 \times [0, T] \to SU(2) \simeq S^3$.

Before we proceed, let us recapitulate the above result using the spaces mentioned earlier: first, the fact that any field $g(x)$ can be characterized by the soliton number shows that, like the configuration space (2.7), the space $\mathcal{G}$ consists of n-soliton sectors $\mathcal{G}_n$ which are disconnected each other,

$$\mathcal{G} = \bigcup_{n \in \mathbb{Z}} \mathcal{G}_n, \tag{3.3}$$

with the integer n being the element of $\pi_0(\mathcal{G}) = \mathbb{Z}$. Second, the fact that gauge transformations preserve each sector $\mathcal{G}_n$ means that

$$\mathcal{Q}_n = \mathcal{G}_n / \mathcal{H}. \tag{3.4}$$

We have learned in sect.2 that multiplication in $\widetilde{\mathcal{G}}$ is well-defined only for the cases (i) – (iii). It is interesting to note that the soliton number becomes additive or subtractive depending on the cases, that is,

$$Q(gg') = \pm\, Q(g) + Q(g'), \tag{3.5}$$

where the '+' sign holds for case (ii) where $g' \in \mathcal{G}^+$, whereas the '−' holds for case (iii) where $g' \in \mathcal{G}^-$. Indeed, from (3.2) we obtain

$$Q(gg') - Q(g') = -\frac{1}{4\pi} \int_{\partial D^2} \mathrm{Tr}\,(g' T_3 g'^{-1})(g^{-1} dg), \tag{3.6}$$

but the r.h.s. turns out to be just $\pm Q(g)$, since from (2.20) we have $g'(x) T_3 g'(x)^{-1} = g'(\infty) T_3 g'(\infty)^{-1} = \pm T_3$, where '+' holds for (ii) and '−' for (iii). For case (i) the r.h.s. of (3.6) vanishes, which shows that the soliton number $Q(g)$ is invariant under global

$SU(2)$ left-transformations (2.22). Thus, in particular, the soliton number is preserved $Q(e^{\pi T_2} g) = Q(g)$ under the parity operation (2.16).

From the property (3.5) it follows that the inverse g^{-1} of an even element $g \in \mathcal{G}^+$ has the opposite soliton charge $Q(g^{-1}) = -Q(g)$, and therefore the configuration g^{-1} may be called 'anti-soliton' with respect to g. On the other hand, the inverse g^{-1} of an odd element $g \in \mathcal{G}^-$ has the same soliton charge $Q(g^{-1}) = Q(g)$, and hence in this case we may take the transpose ${}^{\mathrm{t}}g$ of g to get an anti-soliton, on account of $Q({}^{\mathrm{t}}g) = -Q(g)$. We would need to go through a more involved procedure to deduce these results based on the additive/subtractive property of the soliton number, if we used the original spin vectors $\mathbf{n}(x)$.

Let us next consider how the Hopf number in (1.4) reads in the adjoint orbit parametrization. For this, we first restrict ourselves to the 0-soliton sector $\mathcal{Q}_0$, and consider a boundary condition periodic in time such that $\mathbf{n}(x)$ becomes a constant vector $\mathbf{n}(\infty)$ both at $t = 0$ and T,

$$\mathbf{n}(\mathbf{x}, T) = \mathbf{n}(\mathbf{x}, 0) = \mathbf{n}(\infty). \tag{3.7}$$

This allows us to identify all points on the space S^2 at $t = 0$ and reduce them to a single point (similar reduction can be done for the space S^2 at $t = T$), whereby regard the spacetime under consideration as S^3. Hence, in this case $\mathbf{n}(x)$ provides a map $S^3 \to S^2$, which is known to be classified by the Hopf number which characterizes the homotopy group (2.8). Corresponding to (3.7), we may consider the boundary condition for the field $g(x)$ as

$$g(\mathbf{x}, T) = g(\mathbf{x}, 0)\, h_{\mathrm{c}} = g(\infty), \tag{3.8}$$

where $g(\infty)$ is the element specified in (2.10) and $h_{\mathrm{c}} \in H$ is an arbitrary constant element. Then, a similar reasoning employed for $\mathbf{n}(x)$ allows us to regard that the spacetime for the map $g(x)$ is S^3 and, accordingly, $g(x)$ provides a map $S^3 \to SU(2)$. Upon using the relation (2.1), we find after a little algebra that the Hopf term $H(\mathbf{n})$ in (1.4) becomes local [25] (see also Appendix),

$$H(g) = \frac{1}{48\pi^2} \int_{S^3} \mathrm{Tr}\, (g^{-1}(x) dg(x))^3. \tag{3.9}$$

Since this gives the degree of mapping $S^3 \to SU(2)$, the identity $H(g) = H(\mathbf{n})$ implies the relation between the two integers, $\pi_3(SU(2)) = \mathbb{Z}$ and $\pi_3(S^2) = \mathbb{Z}$.

We now proceed to define the Hopf number to configurations which have nonvanishing soliton numbers. To this end, let us impose the following boundary condition in time,

$$g(\mathbf{x}, T) = g(\mathbf{x}, 0)\, h_{\rm c}, \tag{3.10}$$

namely, we leave out the constancy condition in space imposed in (3.8), as it can be implemented only in the 0-soliton sector. We then consider the configuration,

$$\bar{g}(x) := g(\infty)\, g^{-1}(\mathbf{x}, 0)\, g(\mathbf{x}, t), \tag{3.11}$$

which still lies in the space $\mathcal{G}$ if $g \in \mathcal{G}$. Moreover, we observe that $\bar{g}(x)$ defined in (3.11) has the soliton number zero $\bar{g} \in \mathcal{G}_0$ since it reduces to the identity element at $t = 0$, and that it fulfills the boundary condition (3.8) with $\bar{g}(\infty) = 1$ (which implies $\bar{g} \in \mathcal{G}^+$). It is thus admissible to assign the Hopf number to a configuration $g \in \mathcal{G}_n$ of any soliton number n first by converting it to $\bar{g} \in \mathcal{G}_0$ and then evaluating the value of the Hopf invariant $H(\bar{g})$.

To illustrate our point, let us introduce the coordinates $(\alpha, \beta) \in [0, 2\pi] \times [0, \pi]$ on the disc D^2 by $\mathbf{x} = \left(\frac{\beta}{\pi}\cos\alpha, \frac{\beta}{\pi}\sin\alpha\right)$ with the boundary ∂D^2 being identified with points having $\beta = \pi$, and consider the familiar 1-soliton configuration which evolves dynamically with respect to the collective coordinate $\phi(t)$:

$$\mathbf{n}_1(x) = \big(\cos(\alpha + \phi(t))\sin\beta,\, \sin(\alpha + \phi(t))\sin\beta,\, \cos\beta\big). \tag{3.12}$$

The corresponding expression in the adjoint orbit parametrization is

$$g_1(x) = e^{\phi(t)T_3}\, g_1(\mathbf{x}), \tag{3.13}$$

where $g_1(\mathbf{x})$ stands for the static 1-soliton (Skyrmion) configuration,

$$g_1(\mathbf{x}) = e^{\alpha T_3}\, e^{\beta T_2}\, e^{-\alpha T_3}. \tag{3.14}$$

Note that the 1-soliton configuration $g_1(\mathbf{x})$ belongs to the parity odd subspace $\mathcal{G}^-$ and, since $e^{\phi(t)T_3} \in \mathcal{G}^+$, the configuration $g_1(x)$ in (3.13) also belongs to $\mathcal{G}^-$. We also find that $g_1(x)$ possesses a unit soliton number and further fulfills (3.10) if $\phi(t)$ satisfies a proper periodic boundary condition, say,

$$\phi(0) = 0 \qquad \text{and} \qquad \phi(T) = 2m\pi, \qquad \text{with} \quad m \in \mathbb{Z}. \tag{3.15}$$

Then, according to (3.11) we consider

$$\bar{g}_1(x) = e^{\pi T_2}\, g_1^{-1}(\mathbf{x})\, e^{\phi(t) T_3}\, g_1(\mathbf{x}). \tag{3.16}$$

In order to evaluate the Hopf number for this $\bar{g}_1(x)$, we first change the spacetime from S^3 to $D^2 \times [0, T]$, which can be done by proceeding conversely to what we did before. Although this procedure is in principle unnecessary for a 0-soliton configuration such as $\bar{g}_1(x)$, this allows us to evaluate the integral of the Hopf number for $\bar{g}_1(x)$ from its constituent pieces appearing in the decomposition (3.16) which are defined on $D^2 \times [0, T]$. Indeed, a direct computation gives

$$H(\bar{g}_1) = \frac{1}{2\pi} \int_0^T dt\, \frac{d\phi}{dt} \times Q(g_1) = m, \tag{3.17}$$

which shows that the zero soliton configuration (3.16) just constructed possesses the Hopf number m.

Returning to a generic configuration $\bar{g}(x)$, we observe that, unlike the soliton number, the Hopf number so defined always enjoys the additive property,

$$H(\bar{g}\bar{g}') = H(\bar{g}) + H(\bar{g}'), \tag{3.18}$$

once multiplication is well-defined for any 0-soliton configurations $\bar{g}$, $\bar{g}'$. This can be readily confirmed by substituting $\bar{g}\bar{g}'$ directly in (3.9), which yields

$$H(\bar{g}\bar{g}') - H(\bar{g}) - H(\bar{g}') = -\frac{1}{16\pi^2} \int_{S^3} d\, \mathrm{Tr}\, (\bar{g}^{-1} d\bar{g})(d\bar{g}'\, \bar{g}'^{-1}). \tag{3.19}$$

Then we find that the r.h.s. vanishes identically since S^3 has no boundary, establishing the additivity of the Hopf number. With the help of this additive property, we can check that the Hopf number is invariant under the gauge transformation (2.2) which, for our $\bar{g}(x)$ in (3.11), amounts to

$$\bar{g}(x) \longrightarrow g(\infty)\, h^{-1}(\mathbf{x}, 0)\, g^{-1}(\infty)\, \bar{g}(x)\, h(x). \tag{3.20}$$

Note that the product configuration appearing in the r.h.s. belongs to $\mathcal{G}$ if $g \in \mathcal{G}$, and that it has the soliton number zero because gauge transformations are trivial on the boundary, *i.e.*, $Q(h) = 0$. Then, from $H(h^{-1}(\mathbf{x}, 0)) = H(h(x)) = 0$ we find

$$H(\bar{g}) \longrightarrow H(g(\infty)\, h^{-1}(\mathbf{x}, 0)\, g^{-1}(\infty)\, \bar{g}(x)\, h(x)) = H(\bar{g}), \tag{3.21}$$

as announced.

4. Hopf term as a quantum effect

Having assigned the Hopf number as well as the soliton number to a generic configuration, we now come to the point to show that the Hopf term in the $2+1$ dimensional NLSM can be regarded as a quantum mechanically induced term. More explicitly, we show that in the configuration space $\mathcal{Q}$ there exists a degree of freedom represented by an angle, and that this S^1 degree of freedom is responsible for inducing the Hopf term at the quantum level.

For the sake of brevity, we restrict our attention again to the 0-soliton sector $\mathcal{Q}_0$ where we can use $g(x)$ rather than $\bar{g}(x)$ for the Hopf number. We do not lose generality by this, because any configuration of nonvanishing soliton number can always be brought to a 0-soliton configuration by the conversion procedure (3.11). Now, suppose that the given configuration $g(x) \in \mathcal{G}$ has the Hopf number m. We then consider

$$\hat{g}(x) := g(x)\, g_{\rm st}^{-1}(x), \tag{4.1}$$

where $g_{\rm st}(x) = e^{-\pi T_2}\, \bar{g}_1(x)$ is the 'standard' configuration possessing the Hopf number m and the soliton number 0, constructed with the help of $\bar{g}_1(x)$ in (3.16). Note that since $g_{\rm st}^{-1}$ in (4.1) belongs to the even parity subspace $\mathcal{G}^+$, we have $\hat{g}(x) \in \mathcal{G}$ by (2.14). Making use of the additive property (3.5) of the soliton number valid for case (ii), we obtain

$$Q(\hat{g}(x)) = Q(g(x)) + Q(g_{\rm st}^{-1}(x)) = 0, \tag{4.2}$$

showing that the configuration $\hat{g}(x)$ has zero soliton number. Moreover, using the additivity of the Hopf number (3.18) we find that $\hat{g}(x)$ has zero Hopf number, too. We therefore conclude, by solving (4.1) in favor of $g(x)$, that any 0-soliton configuration $g(x)$ with the Hopf number m can be decomposed into the form,

$$g(x) = \hat{g}(x)\, g_{\rm st}(x). \tag{4.3}$$

In words, any 0-soliton configuration can be written as a product of the specific 0-soliton configuration $g_{\rm st}(x)$ having Hopf number m, and the 'trivial' configuration $\hat{g}(x)$ having vanishing soliton and Hopf numbers. The point to be stressed is that, in the generic configuration of vanishing soliton number, the dynamical degree of freedom that governs the Hopf number is the angle variable $\phi(t)$ in $g_{\rm st}(x)$. As stated above, this is also valid to any configuration $g(x)$ of nonvanishing soliton number, on account of the conversion procedure (3.11).

This result needs to be examined more carefully, since one needs to pay attention to the distinction between the two spaces of maps, the true configuration space $\mathcal{Q}$ in (2.4) and the space of maps $\mathcal{G}$ in (2.11), which are related by (2.13). However, from the fact that gauge transformations (2.2) are right-transformations which do not change the soliton number nor the Hopf number (because these topological numbers are gauge invariant) we learn that the degrees of freedom associated with the gauge transformations represented by $\mathcal{H}$ reside entirely in the last piece $\hat{g}(x)$ in the decomposition (4.3). Stated in terms of spaces, (4.3) implies the decomposition $\mathcal{G}_0 = S^1 \times \widehat{\mathcal{G}}_0$, where $\widehat{\mathcal{G}}_0$ is the space of trivial configurations $\hat{g}(x)$ for which $\pi_1(\widehat{\mathcal{G}}_0) = 0$. Thus, combining the gauge invariance of the S^1 degrees of freedom together with (3.4) for $n = 0$, we find

$$\mathcal{Q}_0 = S^1 \times \big(\widehat{\mathcal{G}}_0/\mathcal{H}\big). \tag{4.4}$$

Since a similar decomposition is valid to any soliton sector $\mathcal{Q}_n$, we conclude that

$$\mathcal{Q} = S^1 \times \big(\widehat{\mathcal{G}}/\mathcal{H}\big), \tag{4.5}$$

where $\widehat{\mathcal{G}} = \bigcup_n \widehat{\mathcal{G}}_n$. In short, the configuration space $\mathcal{Q}$ admits a decomposition in such a way that the gauge invariant angle degree of freedom represented by the circle S^1 is factored out.

Now, let us consider the quantization of the NLSM starting with the classical action (1.1). This, however, is a formidable task to do in practice, because the configuration space $\mathcal{Q}$ is nowhere close in structure to a Euclidean space, for which we know how to quantize by, *e.g.*, the canonical quantization programme. However, we also know the consequence of quantization on relatively simple but nontrivial spaces, such as coset spaces G/H by, *e.g.*, Mackey's quantization procedure [26]. In particular, when the configuration space is a circle S^1, we know from Mackey's quantization procedure or Schulman's analysis of path-integral quantization [22] that, at the quantum level, an induced term arises in the form (1.5) with parameter θ which labels the superselection sectors of the quantum theory.

This implies that, when we focus on the quantization of the S^1 degree of freedom in the configuration space $\mathcal{Q}$ given by the angle $\phi(t)$, we obtain precisely the induced term (1.5) as a quantum effect. Then, from the relation (3.17) we obtain

$$I_{\text{top}} = \theta\, H(g_{\text{st}}) = \theta\, H(\bar{g}) = \frac{\theta}{48\pi^2} \int_{S^3} \mathrm{Tr}\, (\bar{g}^{-1}(x) d\bar{g}(x))^3. \tag{4.6}$$

This shows that the Hopf term $H(\bar{g})$ — rather than $H(g)$ which is ill-defined for configurations of nonvanishing soliton numbers — can be acquired as an induced term, purely from quantization.

5. Discussions and outlooks

We have seen in this paper that the Hopf term in the $O(3)$ NLSM in $2+1$ dimensions can be induced quantum mechanically as a manifestation of the (inevitable) ambiguity in quantizing the S^1 degree of freedom of the configuration space. The tool we employed throughout our argument is the adjoint orbit parametrization (2.1), which not only renders the Hopf term local, as does the commonly used $\mathbb{C}P^1$ parametrization, but also is very useful in dealing with configurations of nonvanishing topological (soliton and/or Hopf) numbers. In particular, it was shown that the additivity of those topological numbers, (3.5) and (3.18), which is available in the adjoint orbit parametrization, allows decomposing any given configuration into constituent configurations of certain topological numbers. This enables us to extract a specific degree of freedom, which in our case is S^1, in the true configuration space $\mathcal{Q}$. It should be noted that the S^1 degree of freedom is extracted from a generic configuration, rather than being introduced as a collective coordinate around the 1-soliton configuration (3.12) as often done in, *e.g.*, refs.[16,27]. It should also be stressed that the Hopf number can be assigned to sectors of nonvanishing soliton numbers only after the conversion to the vanishing soliton sector is performed. To our knowledge, this conversion has not been discussed before, possibly because with spin vectors $\mathbf{n}(x)$ the procedure would become quite involved.

Our investigation on the quantum mechanical aspect of the topological structure of the space $\mathcal{Q}$ in the NLSM is a continuation of a study in the Abelian sigma model in $1+1$ dimensions [28]. In both of the models, we extracted an S^1 degree of freedom from the configuration space after decomposition, and found that the quantum effect associated with the S^1 does induce the topological term as anticipated in the path-integral quantization [5,23]. This result reinforces the view that the induced terms appearing in these models are intrincically of quantum origin, just like the θ-term in QCD. We need to note, however, that the investigation carried out thus far is still a primitive one, since it is concerned only with the fundamental group $\pi_1(\mathcal{Q})$ of the configuration space, and in this sense it would be natural to expect more to occur at the quantum level from the topological structure associated with higher homotopy groups such as, in the case of the NLSM, $\pi_2(\mathcal{Q}) = \mathbb{Z}_2$,

$\pi_3(\mathcal{Q}) = \mathbb{Z}_2$ and so forth. Indeed, it is known [29,30] that there arises an induced term given by the Dirac monopole potential when one quantizes on a sphere $\mathcal{Q} = S^2$ for which $\pi_2(\mathcal{Q}) = \mathbb{Z}$, and further, that a similar phenomenon occurs on S^n in general (see also [31]). Although it is not yet clear how to realize such a phenomenon in the NLSM, we expect that our direct method of extracting the crucial degree(s) of freedom from the space $\mathcal{Q}$ is more fruitful than the phase ambiguity consideration in the path-integral [5,23,21].

In this respect, it might be worth mentioning the same NLSM placed in $1+1$ dimensions. In this case, if one proceeds analogously to the $2+1$ dimensional case assuming the boundary condition on the spin vector field similarly to (2.3) *etc.*, the configuration space will be $\mathcal{Q} = \mathrm{Map}_0(S^1, S^2)$. Then, using an argument similar to that used for the $2+1$ dimensional case, one obtains $\pi_0(\mathcal{Q}) = \pi_1(S^2) = 0$, which shows that the configuration space is connected and hence has no distinct soliton sectors. On the other hand, one also observes that $\pi_1(\mathcal{Q}) = \pi_2(S^2) = \mathbb{Z}$, which indicates that the space $\mathcal{Q}$ is multiply connected, implying that there exist configurations possessing topological (winding) numbers dynamically, bearing the term (1.2). One further notices that $\pi_2(\mathcal{Q}) = \pi_3(S^2) = \mathbb{Z}$, whose quantum mechanical consequence may appear as an induced potential of the Dirac monopole as mentioned earlier. The adjoint orbit parametrization will continue to be useful in studying this case, although one needs to take care of the maps $g(x)$ anew in regard to the topology of the configuration space, as we did for the $2+1$ dimensional case in the present paper.

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Appendix

The purpose of this Appendix is to put in order some of the basic mathematical ingredients of the $2+1$ dimensional $O(3)$ NLSM, *i.e.*, mappings and topological invariants (which are at times handled loosely in the text) based on more rigorous definitions (see, *e.g.*, [32]). To this end, we use different notations from those used in the text in order to distinguish the various mappings strictly, while in the main text we preferred to use simpler notations. We also give a proof of the identity (2.5) for homotopy groups.

Soliton number: We begin by defining the soliton number in the NLSM. By the canonical embedding $S^2 \hookrightarrow \mathbb{R}^3$ a point of S^2 can be identified with a unit vector $\mathbf{n} \in \mathbb{R}^3$. The 2-form

$$\omega := \frac{1}{8\pi}\mathbf{n} \cdot (d\mathbf{n} \times d\mathbf{n}) \tag{A.1}$$

on S^2 is closed and has the integral $\int_{S^2} \omega = 1$, and hence is a generator of the de Rham cohomology $H^2_{\mathrm{DR}}(S^2)$. With the coordinates $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ it reads $\omega = \dfrac{1}{4\pi}\sin\theta d\theta \wedge d\phi$. The $O(3)$ NLSM is described by the field variable $\varphi : M \to S^2$, where M is the base spacetime, which in this paper is assumed to be $M = S^2 \times [0, T]$. Restriction to a particular time $t \in [0, T]$ provides the configuration $\varphi_t : S^2 \times \{t\} \to S^2$ on a time-slice in M, where the former S^2 represents the base space and the latter S^2 is the target space. Then the pullback of ω by φ gives a closed 2-form $J := \varphi^*\omega$ on M, which is called topological current whose integration over the fixed time-slice yields an integer,

$$Q(\varphi_t) := \int_{S^2} \varphi_t^*\omega. \tag{A.2}$$

In fact, $Q(\varphi_t)$ counts the wrapping number of the mapping $\varphi_t : S^2 \to S^2$. It therefore is independent of time t giving the integers related to the homotopy group $\pi_2(S^2) = \mathbb{Z}$, and hence can be used to define the 'soliton number' to the field φ as a topological charge.

Hopf invariant: There is another functional to characterize the homotopy group $\pi_3(S^2) = \mathbb{Z}$, that is, the Hopf invariant. It is defined as follows. Assume that the field $\varphi : S^2 \times [0, T] \to S^2$ becomes constant at end times, *i.e.*, both $\varphi_0 : S^2 \times \{0\} \to S^2$ and $\varphi_T : S^2 \times \{T\} \to S^2$ are constant mappings. Then it can be regarded as a mapping $\varphi : S^3 \to S^2$. Take a 1-form A on S^3 which satisfies the equation,

$$dA = J = \varphi^*\omega. \tag{A.3}$$

Note that a solution for A exists, since $H^2_{\mathrm{DR}}(S^3) = 0$ implies that the closed 2-form $\varphi^*\omega$ is exact on S^3. The solution is of course not unique, because if A is a solution, $A + df$ is also a solution for any function f on S^3. However, the Hopf invariant (which we also refer to as the Hopf number) defined by

$$H(\varphi) := -\int_{S^3} A \wedge J, \tag{A.4}$$

is free from the arbitrariness thanks to the closedness (current conservation) $dJ = 0$.

Hopf fibration by the adjoint orbit parametrization: The topological nature of the Hopf invariant becomes transparent by considering the Hopf fibration. Let $T_a = \dfrac{\sigma_a}{2i}$ ($a = 1, 2, 3$) be a basis of the Lie algebra $\mathfrak{su}(2)$, with which the inner product $\langle X, Y\rangle := -2\ \mathrm{tr}\ (XY)$ for $X, Y \in \mathfrak{su}(2)$ is equipped. Then, using the isomorphism between $\mathfrak{su}(2)$ and $\mathbb{R}^3$ provided by the identification of $X = \sum_a X_a T_a \in \mathfrak{su}(2)$ and $\mathbf{X} = (X_1, X_2, X_3) \in \mathbb{R}^3$, we furnish the Hopf fibration $p : S^3 \to S^2$ with fibre S^1 using the mapping given by the adjoint action on a fixed element, say, T_3 of $\mathfrak{su}(2)$ as $p : g \mapsto \mathrm{Ad}(g)T_3 = gT_3g^{-1}$. The fibre is identified with the subgroup $U(1)$ of $SU(2)$ generated by T_3.

If we pullback the 2-form ω on S^2 to $SU(2) \simeq S^3$ by p, then again by $H^2_{\mathrm{DR}}(S^3) = 0$ the closed 2-form $p^*\omega$ is exact on the S^3, and therefore there exists a 1-form ρ on S^3 such that $p^*\omega = d\rho$. If we put $g^{-1}dg = \sum_a \gamma_a T_a$, we find

$$p^*\omega = \frac{1}{8\pi}\langle gT_3g^{-1}, [\, d(gT_3g^{-1}), d(gT_3g^{-1})\,]\rangle = \frac{1}{4\pi}\gamma_1 \wedge \gamma_2 = -\frac{1}{4\pi}d\gamma_3, \tag{A.5}$$

which shows that the solution is given by the 1-form $\rho = -\dfrac{1}{4\pi}\gamma_3 = -\dfrac{1}{4\pi}\langle T_3, g^{-1}dg\rangle$ up to an exact 1-form.

Suppose that, given $\varphi : S^3 \to S^2$, there exists a mapping $\tilde\varphi : S^3 \to S^3$ satisfying the following commutative diagram,

$$\begin{array}{ccc} & & S^3 \\ & \tilde\varphi \nearrow & \big\downarrow p \\ S^3 & \xrightarrow[\varphi]{} & S^2 \end{array} \tag{A.6}$$

Then we see that the Hopf invariant gives just the degree of mapping $\tilde\varphi : S^3 \to S^3$ (from the base space S^3 to the target S^3). Indeed, since $p \circ \tilde\varphi = \varphi$, we have

$$dA = J = \varphi^*\omega = (p \circ \tilde\varphi)^*\omega = \tilde\varphi^*(p^*\omega) = \tilde\varphi^*(d\rho) = d(\tilde\varphi^*\rho), \tag{A.7}$$

which shows that $A = \tilde\varphi^*\rho = -\dfrac{1}{4\pi}\tilde\varphi^*\gamma_3$ is a solution for (A.3). When this is combined with $J = \dfrac{1}{4\pi}\tilde\varphi^*(\gamma_1 \wedge \gamma_2)$ which is obtained from (A.5), we find that the Hopf invariant (A.4) reads

$$\begin{aligned} H(\varphi) &= \frac{1}{(4\pi)^2}\int_{S^3} \tilde\varphi^*(\gamma_1 \wedge \gamma_2 \wedge \gamma_3) \\ &= \frac{1}{6(4\pi)^2}\int_{S^3} \tilde\varphi^*\langle g^{-1}dg, [\, g^{-1}dg, g^{-1}dg\,]\rangle. \end{aligned} \tag{A.8}$$

The fact that the 3-form $\omega^{(3)}$, which is defined by writing $H = \int_{S^3} \tilde\varphi^*\omega^{(3)}$, gives a generator of $H^3_{\mathrm{DR}}(S^3)$ can be seen explicitly by introducing the coordinates on $SU(2)$ by $(\phi, \theta, \psi) \in$

$[\,0, 2\pi\,] \times [\,0, \pi\,] \times [\,0, 4\pi\,]$ and put the element $g \in SU(2)$ as $g = e^{\phi T_3} e^{\theta T_2} e^{\psi T_3}$. This leads to

$$\begin{aligned} g^{-1} dg = {} & (\sin\psi d\theta - \cos\psi \sin\theta d\phi) T_1 \\ & + (\cos\psi d\theta + \sin\psi \sin\theta d\phi) T_2 + (\cos\theta d\phi + d\psi) T_3, \end{aligned} \tag{A.9}$$

and hence

$$\int_{SU(2)} \omega^{(3)} = \int_{SU(2)} \sin\theta d\theta \wedge d\phi \wedge d\psi = 1, \tag{A.10}$$

as required. We recall that the spectral sequence associated with the exact sequence $S^1 \to S^3 \to S^2$ provides the exact sequence of homotopy groups,

$$\cdots \to \pi_3(S^1) \to \pi_3(S^3) \to \pi_3(S^2) \to \pi_2(S^1) \to \cdots \tag{A.11}$$

Since $\pi_3(S^1) = \pi_2(S^1) = 0$, we have $\pi_3(S^3) = \pi_3(S^2) = \mathbb{Z}$. The correspondence between $\tilde{\varphi}$ and φ realizes this isomorphism between $\pi_3(S^3)$ and $\pi_3(S^2)$.

Existence of the map $\tilde{\varphi}$*:* The foregoing statement for the Hopf invariant rests entirely on the assumption that the mapping $\tilde{\varphi}$ exists, which is nontrivial. Before answering to the question whether such a mapping really exists or not, let us first consider the static version of the problem, that is, the question whether the map $\tilde{\sigma} : S^2 \to S^3$ satisfying the commutative diagram,

$$\begin{array}{ccc} & & S^3 \\ & \tilde{\sigma} \nearrow & \big\downarrow p \\ S^2 & \underset{\sigma}{\longrightarrow} & S^2 \end{array} \tag{A.12}$$

exists to a given static configuration $\sigma : S^2 \to S^2$. Below we shall show that the necessary and sufficient condition for the existence of $\tilde{\sigma}$ is that the soliton number vanishes $Q(\sigma) = 0$. The necessity is easy; if we assume $\tilde{\sigma}$ to exist, then by definition we find

$$Q(\sigma) = \int_{S^2} \sigma^* \omega = \int_{S^2} (p \circ \tilde{\sigma})^* \omega = \int_{S^2} \tilde{\sigma}^* (p^* \omega) = \int_{S^2} \tilde{\sigma}^* (d\rho) = \int_{S^2} d(\tilde{\sigma}^* \rho) = 0. \tag{A.13}$$

To show the sufficiency, let D^2 denote a two-dimensional disk of unit radius $D^2 := \{\mathbf{x} = (x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$, and consider its one-point compactification by

$$\begin{array}{cccc} c: & D^2(\subset \mathbb{R}^2) & \to & S^2(\subset \mathbb{R}^3), \\ & \left(\dfrac{\theta}{\pi}\cos\phi, \dfrac{\theta}{\pi}\sin\phi\right) & \mapsto & (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \end{array} \tag{A.14}$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. The mapping c brings all points on the boundary ∂D^2 to the South pole $\mathbf{x}_{\rm S} = (0,0,-1) \in S^2$. Then, since D^2 is contractible, for any $\sigma : S^2 \to S^2$ there exists $\hat{\sigma} : D^2 \to S^3$ which satisfies the commutative diagram,

$$\begin{tikzcd} D^2 \arrow[r, "\hat{\sigma}"] \arrow[d, "c"'] & S^3 \arrow[d, "p"] \\ S^2 \arrow[r, "\sigma"'] & S^2 \end{tikzcd} \tag{A.15}$$

The mapping $\hat{\sigma}$ is not uniquely determined, since it can be replaced by the gauge transformed one $\hat{\sigma}' = \hat{\sigma} \cdot h$ with $h : D^2 \to U(1)$. However, this ambiguity does not affect the soliton number $Q(\sigma)$ given by

$$Q(\sigma) = \int_{D^2} d(\hat{\sigma}^* \rho) = \int_{\partial D^2} \hat{\sigma}^* \rho = -\frac{1}{4\pi} \int_{\partial D^2} \hat{\sigma}^* \langle T_3, g^{-1} dg \rangle. \tag{A.16}$$

Clearly, the soliton number $Q(\sigma)$ in (A.16) counts the degree of the restricted mapping $\hat{\sigma}| : \partial D^2 \simeq S^1 \to p^{-1}(\sigma(\mathbf{x}_{\rm S})) \simeq S^1$. Thus, if $Q(\sigma) = 0$ then $\hat{\sigma}$ admits a gauge transformation to a mapping $\hat{\sigma}' : D^2 \to S^3$ which is constant along the boundary ∂D^2. Hence in this case we can shrink ∂D^2 to a point to get a mapping $\tilde{\sigma}$ satisfying the diagram,

$$\begin{tikzcd} D^2 \arrow[r, "\hat{\sigma}'"] \arrow[d, "c"'] & S^3 \arrow[d, "p"] \\ S^2 \arrow[r, "\sigma"'] \arrow[ur, "\tilde{\sigma}"] & S^2 \end{tikzcd} \tag{A.17}$$

as we wanted.

Now, to answer the question of the existence of the mapping $\tilde{\varphi}$ satisfying (A.6), we recall that $\varphi : S^3 \to S^2$ is regarded as $\varphi : S^2 \times [\,0, T\,] \to S^2$ which takes constant values at end times of the period $[\,0, T\,]$. But for this to be the case the mapping $\varphi_0 : S^2 \to S^2$ must have 0-soliton number and hence φ_t also has $Q(\varphi_t) = 0$ for any $t \in [\,0, T\,]$. Consequently, the above argument shows that there indeed exists a map $\tilde{\varphi}_t$ and hence $\tilde{\varphi}$ satisfying (A.6), if $Q(\varphi_t) = 0$.

On the other hand, this also shows that, if $Q(\varphi_t) \neq 0$ for $\varphi : S^2 \times [\,0, T\,] \to S^2$, then there cannot exist a mapping $\tilde{\varphi} : S^2 \times [\,0, T\,] \to S^3$ satisfying $p \circ \tilde{\varphi} = \varphi$. However, since the mapping $\hat{\sigma}$ in diagram (A.15) always exists, the mapping $\hat{\varphi}$ satisfying

$$\begin{tikzcd} D^2 \times [\,0, T\,] \arrow[r, "\hat{\varphi}"] \arrow[d, "c"'] & S^3 \arrow[d, "p"] \\ S^2 \times [\,0, T\,] \arrow[r, "\varphi"'] & S^2 \end{tikzcd} \tag{A.18}$$

is also guaranteed to exist. The configuration $g(x)$ used for the adjoint parametrization to describe a generic (nonvanishing) soliton sector in the text is this mapping $\hat{\varphi}$.

Configurations of nontrivial topological numbers: Configurations possessing a nonvanishing soliton or Hopf number may be furnished explicitly as follows. Take some integer $n \in \mathbb{Z}$ and consider the mapping,

$$\begin{array}{cccc} \sigma_n : & S^2 & \rightarrow & S^2 \\ & (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) & \mapsto & (\sin\theta\cos n\phi, \sin\theta\sin n\phi, \cos\theta). \end{array} \tag{A.19}$$

The lift $\hat{\sigma}_n : D^2 \rightarrow S^3$ defined in (A.15) is given by $\hat{\sigma}_n(\mathbf{x}) = e^{n\phi T_3} e^{\theta T_2} e^{-n\phi T_3}$. Then it is readily confirmed from (A.16) that the configuration has soliton number n,

$$Q(\sigma_n) = -\frac{1}{4\pi}\int_{\partial D^2} \hat{\sigma}_n^* \gamma_3 = -\frac{1}{4\pi}\int_{S^1} (-2n d\phi) = n. \tag{A.20}$$

We may next use this configuration to construct a configuration which has a nonvanishing Hopf number. This can be achieved by taking the mapping $\hat{\tau}_m : [\,0, T\,] \rightarrow S^1 (\subset S^3)$ given by $\hat{\tau}_m(t) = e^{2\pi m (t/T)\, T_3}$ and thereby construct the mapping $\hat{\varphi} : D^2 \times [\,0, T\,] \rightarrow S^3$ by $\hat{\varphi}(\mathbf{x}, t) := \hat{\sigma}_n^{-1}(\mathbf{x}) \hat{\tau}_m(t) \hat{\sigma}_n(\mathbf{x})$. It is then straightforward to show that

$$\langle \hat{\varphi}^{-1} d\hat{\varphi}, [\,\hat{\varphi}^{-1} d\hat{\varphi}, \hat{\varphi}^{-1} d\hat{\varphi}\,] \rangle = 24\pi\, m \,\frac{dt}{T}\, \langle T_3, d(d\hat{\sigma}_n \hat{\sigma}_n^{-1}) \rangle, \tag{A.21}$$

which leads to the Hopf number,

$$H(\varphi) = \frac{4\pi}{(4\pi)^2} \int_0^T m\, \frac{dt}{T} \int_{\partial D^2} \langle T_3, d\hat{\sigma}_n \hat{\sigma}_n^{-1} \rangle = m\, n. \tag{A.22}$$

The case $n = 1$ provides a configuration having a unit Hopf number $H = m$ (with $Q = 0$), and this is the standard configuration $g_{\rm st}(x)$ used in the text.

Proof of the homotopy identity (2.5): Finally, for completeness we provide a proof[5] of the identity (2.5) for homotopy groups. First we note that for compact spaces X, Y and Z, we have the homeomorphism [33] between the spaces of based maps,

$$\mathrm{Map}_0\big(X,\, \mathrm{Map}_0(Y, Z)\big) = \mathrm{Map}_0(X \wedge Y, Z), \tag{A.23}$$

where the wedge stands for the smash product of the two spaces. Take $X = S^k$, $Y = S^n$ and $Z = S^m$ with nonnegative intergers k, n and m, and recall the relation $S^k \wedge S^n = S^{k+n}$ valid for spheres [34]. Then it follows that

$$\mathrm{Map}_0\big(S^k,\, \mathrm{Map}_0(S^n, S^m)\big) = \mathrm{Map}_0(S^{k+n}, S^m). \tag{A.24}$$

The identity (2.5) is obtained by considering the homotopy classes for the pointed space,

$$\pi_k\big(\mathrm{Map}_0(S^n, S^m)\big) = \pi_{k+n}(S^m), \tag{A.25}$$

and setting $n = m = 2$.

[5] We owe this to L. Fehér.

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